

Another Kannan Version of Suzuki Fixed Point Theorem

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Abstract: This research paper is inspired from an interesting result relating to fixed point theory of complete metric space. The fixed point theorem by Suzuki characterizes the metric completeness of the underlying space. Suzuki in his further work along with Kikkawa also proved a Kannan version of the same theorem. In this research paper we have proved another Kannan version of the Suzuki theorem.

Keywords: Complete Metric Space, Fixed point.

I. INTRODUCTION

Banach Contraction Principle [1] is very useful because it is a strong tool in nonlinear analysis. It has various generalizations [2 - 14]. But Banach theorem does not characterize the metric completeness. Connell [15] gave an example of a metric space X such that X is not complete and every contraction on X has a fixed point. In this context Kannan fixed point theorem [16] is very important because Subrahmanyam [17] proved that Kannan theorem characterizes the metric completeness of underlying spaces. Tomonari Suzuki [18] also introduced a new type of mapping. This mapping not only generalizes the Banach Contraction Mapping Principle but also characterizes the completeness of the underlying metric space. M. Kikkawa and T. Suzuki [19] presented Kannan version of Suzuki theorem. In this paper we state and prove a fixed point theorem that uses Suzuki mapping and Kannan type of contraction.

II. PRELIMINARIES AND DEFINITIONS

Definition 2.1: Let (X, d) be a metric space and let T be a mapping on X . Then T is called a “Contraction” if there exists $r \in [0, 1)$ such that

$$d(Tx, Ty) \leq rd(x, y) \forall x, y \in X$$

Definition 2.2: Let (X, d) be a metric space and let T be a mapping on X . Then T is called “Kannan” if there exists $\alpha \in [0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \forall x, y \in X.$$

The following famous theorem is referred to as the Banach contraction principle.

Theorem 2.1 (Banach) [1]: Let (X, d) be a complete metric space and let T be a contraction on X . Then T has a unique fixed point.

Theorem 2.2 (Suzuki) [18]: Define a non-increasing function θ from $[0, 1)$ to $(1/2, 1]$ by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1 \end{cases}$$

Then for a metric space (X, d) , the following are equivalent:

- (1) X is complete.
- (2) Every mapping T on X satisfying the following has a fixed point: there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ Implies } d(Tx, Ty) \leq rd(x, y) \text{ for all } x, y \in X.$$

Theorem 2.3 (Kikkawa and Suzuki) [19]: Let T be a mapping on complete metric space (X, d) and θ be a non-increasing function from $[0, 1)$ onto $(1/2, 1]$ defined by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1 \end{cases}$$

Suppose that there exists $r \in [0,1)$ such that

$$\theta(r)d(x,Tx) \leq d(x, y) \text{ Implies}$$

$$d(Tx, Ty) \leq r \max d(x, Tx), d(y, Ty) \text{ for all}$$

$$x, y \in X .$$

Then T has a unique fixed point z and $\lim_{n \rightarrow \infty} T^n x = z$ holds for every $x \in X$.

The following is a Kannan version of the Suzuki Theorem.

Theorem 2.4(Kikkawa and Suzuki) [19]: Let T be a mapping on complete metric space (X, d) and θ be a non-increasing function from $[0,1)$ into $(1/2, 1]$ defined by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1 \end{cases}$$

Let $\alpha \in [0, 1/2)$ and $r = \frac{\alpha}{1-\alpha} \in [0, 1)$.

Suppose that $\theta(r)d(x, Tx) \leq d(x, y)$ implies

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \text{ for all } x, y \in X$$

Then T has a unique fixed point z and $\lim_{n \rightarrow \infty} T^n x = z$ holds for every $x \in X$.

Main Result

The generalization of theorem 2.4 is as follows.

Theorem 3.1: Let (X, d) be a metric space and let T be a mapping on X . Let $x \in X$ satisfy

$$d(Tx, T^2x) \leq rd(x, Tx) \text{ for some } r \in [0, 1) .$$

Then for $y \in X$ either $\frac{1}{1+r} d(x, Tx) \leq d(x, y)$ or

$$\frac{1}{1+r} d(Tx, T^2x) \leq d(Tx, y) \text{ holds.}$$

Proof: We assume $\frac{1}{1+r} d(x, Tx) > d(x, y)$ and

$$\frac{1}{1+r} d(Tx, T^2x) > d(Tx, y) .$$

Then we have

$$\begin{aligned} d(x, Tx) &\leq d(x, y) + d(y, Tx) \\ &< \frac{1}{1+r} [d(x, Tx) + d(Tx, T^2x)] \\ &< \frac{1}{1+r} [d(x, Tx) + rd(x, Tx)] \\ &= \frac{1}{1+r} (1+r)d(x, Tx) = d(x, Tx) \end{aligned}$$

This is a contraction.

Lemma 3.1: The functions $f(x) = \frac{2x^3 + x^2 + x}{1+x}$ and

$g(x) = \frac{2x^3 + x}{1+x}$ are monotonically increasing in the interval

$[0, \infty)$. Also $f(x) < 1$ if $0 \leq x \leq \frac{\sqrt{5}-1}{2}$ and $g(x) < 1$ if

$\frac{\sqrt{5}-1}{2} \leq x \leq \frac{1}{\sqrt{2}}$. This has been illustrated in figure 3.1.

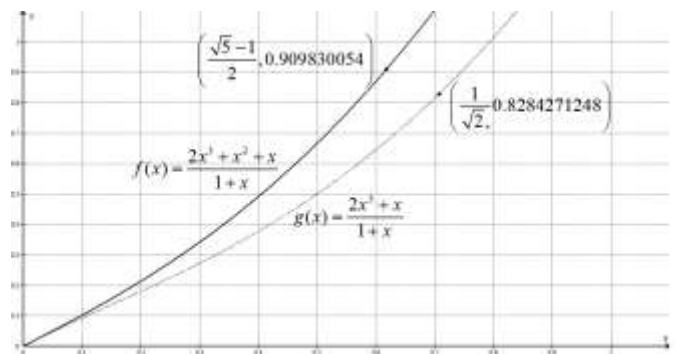


Figure 3.1: Graphs of the functions $f(x)$ and $g(x)$ in the interval $[0, 1]$.

We generalize the theorem 2.4 by using function θ in the interval $[0,1)$ onto $(1/2, 1]$ as follows.

Theorem 3.2: Let (X, d) be a complete metric space and let T be a mapping on X . Define a non-increasing function θ from $[0,1)$ into $(1/2, 1]$ by

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \leq r \leq \frac{\sqrt{5}-1}{2}, \\ \frac{1-r}{r^2}, & \text{if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \leq r < 1 \end{cases}$$

Let $\alpha \in [0, 1/2)$ and $r = \frac{\alpha}{1-\alpha} \in [0, 1)$.

Assume that $\theta(r)d(x, Tx) \leq d(x, y)$ implies

$$d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \text{ for all } x, y \in X .$$

Then T has a unique fixed point z and $\lim_{n \rightarrow \infty} T^n x = z$ holds for every $x \in X$.

Proof: We have

$$\theta(r)d(x, Tx) \leq d(x, Tx), \forall x \in X \quad (\text{Q } \theta(r) \leq 1) .$$

By hypothesis we have

$$d(Tx, T^2x) \leq \alpha d(x, Tx) + \alpha d(Tx, T^2x)$$

$$\therefore d(Tx, T^2x) \leq \frac{\alpha}{1-\alpha} d(x, Tx)$$

$$\therefore d(Tx, T^2x) \leq rd(x, Tx), \forall x \in X.$$

In general

$$d(T^n x, T^{n+1} x) \leq r^n d(x, Tx) \quad \forall x \in X \quad (1)$$

We now fix an element $u \in X$ and define a sequence u_n in X by $u_n = T^n u$.

Then from (1) we have,

$$d(u_n, u_{n+1}) = d(T^n u, T^{n+1} u) \leq r^n d(u, Tu).$$

Taking limit as $n \rightarrow \infty$, since $r < 1$, we get $d(u_n, u_{n+1}) \rightarrow 0$ as $n \rightarrow \infty$.

Thus u_n is a Cauchy sequence in X .

Since X is a complete metric space, u_n converges to some point $z \in X$.

We next show

$$d(Tx, z) \leq rd(x, z), \forall x \in X \setminus z \quad (2)$$

For any $x \in X \setminus z$, we have

$$d(u_n, z) \leq \frac{d(x, z)}{3}, \forall n \geq \nu \text{ for some } \nu \in \mathbb{N}.$$

Then we must have

$$\begin{aligned} \theta(r)d(u_n, Tu_n) &\leq d(u_n, Tu_n) = d(u_n, u_{n+1}) \\ &\leq d(u_n, z) + d(u_{n+1}, z) \\ &\leq \frac{d(x, z)}{3} + \frac{d(x, z)}{3}, \forall n \geq \nu \\ &= \frac{2}{3}d(x, z) = d(x, z) - \frac{d(x, z)}{3} \\ &\leq d(x, z) - d(u_n, z) \leq d(u_n, x) \end{aligned}$$

Thus $\theta(r)d(u_n, Tu_n) \leq d(u_n, x)$.

By hypothesis we then get

$$d(Tu_n, Tx) \leq \alpha d(u_n, Tu_n) + \alpha d(x, Tx), \text{ for } n \geq \nu \quad (3)$$

Taking n tend to ∞ we get

$$\begin{aligned} d(z, Tx) &\leq \alpha d(z, z) + \alpha d(x, Tx) = \alpha d(x, Tx) \\ \therefore d(z, Tx) &\leq \alpha d(x, Tx) \leq \alpha (d(x, z) + d(Tx, z)) \text{ (Triangle inequality)} \end{aligned}$$

Thus

$$d(Tx, z) \leq \alpha d(x, z) + \alpha d(Tx, z)$$

$$\therefore (1-\alpha)d(Tx, z) \leq \alpha d(x, z)$$

$$\therefore d(Tx, z) \leq \frac{\alpha}{1-\alpha} d(x, z)$$

$$\therefore d(Tx, z) \leq rd(x, z)$$

Thus we have shown(2).

Next, we shall show that there exists a $j \in \mathbb{N}$ such that $T^j z = z$

This we show by contradiction method. We assume that $T^j z \neq z$ for all $j \in \mathbb{N}$. As $T^j z \neq z, \forall j \in \mathbb{N}$ we can use inequality(2).

Using inequality (2) we get,

$$d(T^2 z, z) = d(T \circ Tz, z) \leq rd(Tz, z) \quad (4)$$

$$\begin{aligned} d(T^3 z, z) &= d(T \circ T^2 z, z) \leq rd(T^2 z, z) \\ &\leq r^2 d(Tz, z) \end{aligned} \quad (5)$$

$$\begin{aligned} d(T^4 z, z) &= d(T \circ T^3 z, z) \leq rd(T^3 z, z) \\ &\leq r^3 d(Tz, z) \end{aligned}$$

Thus in general

$$d(T^{j+1} z, z) \leq r^j d(Tz, z) \text{ for any } j \in \mathbb{N} \quad (6)$$

We also observe that for any $x \in X \setminus z$,

$$\begin{aligned} d(Tx, z) &= \lim_{n \rightarrow \infty} d(Tx, u_{n+1}) \\ &= \lim_{n \rightarrow \infty} d(Tx, Tu_n) \\ &\leq \lim_{n \rightarrow \infty} \alpha d(u_n, Tu_n) + \alpha d(x, Tx) \text{ (by (3))} \\ &= \alpha d(x, Tx) \end{aligned}$$

Thus

$$d(z, Tx) \leq \alpha d(x, Tx) \quad (7)$$

Therefore

$$\begin{aligned} d(T^{j+1} z, z) &= d(T \circ T^j z, z) \\ &\leq \alpha d(T^j z, T^{j+1} z) \text{ (by (7))} \\ &\leq \alpha r^j d(z, Tz) \text{ (by (1))} \end{aligned} \quad (8)$$

Now we consider the following three cases.

$$\text{Case 1. } 0 \leq r \leq \frac{\sqrt{5}-1}{2}$$

In this case we observe that

$$r \leq \frac{\sqrt{5}-1}{2} \Rightarrow r^2 \leq \left(\frac{\sqrt{5}-1}{2} \right)^2$$

Therefore

$$r^2 + r \leq \left(\frac{\sqrt{5}-1}{2} \right)^2 + \frac{\sqrt{5}-1}{2} = \frac{5-2\sqrt{5}+1+2\sqrt{5}-2}{4} = 1.$$

Thus $r^2 + r \leq 1$.

Assume $d(T^2 z, z) < d(T^2 z, T^3 z)$.

Then we have

$$\begin{aligned}
 d(z, Tz) &\leq d(z, T^2z) + d(Tz, T^2z) \quad (\text{by triangle inequality}) \\
 &< d(T^2z, T^3z) + rd(z, Tz) \quad (\text{by above assumption and (1)}) \\
 &\leq r^2d(z, Tz) + rd(z, Tz) \quad (\text{by (1)}) \\
 &= (r^2 + r)d(z, Tz) \\
 &\leq d(z, Tz) \quad (\text{Q } r^2 + r \leq 1)
 \end{aligned}$$

Thus $d(z, Tz) < d(z, Tz)$. This is a contradiction.

So we have

$$\begin{aligned}
 d(T^2z, z) &\geq d(T^2z, T^3z) = \theta(r)d(T^2z, T^3z) \\
 (\text{Q } \theta(r) = 1 \text{ in this case})
 \end{aligned}$$

By hypothesis

$$d(T^3z, Tz) \leq \alpha d(T^2z, T^3z) + \alpha d(z, Tz) \quad (9)$$

Now consider

$$\begin{aligned}
 d(z, Tz) &\leq d(z, T^3z) + d(T^3z, Tz) \quad (\text{Triangle inequality}) \\
 &\leq r^2d(z, Tz) + \alpha d(T^2z, T^3z) + \alpha d(z, Tz) \quad (\text{by (6) and (9)}) \\
 &\leq r^2d(z, Tz) + \alpha r^2d(z, Tz) + \alpha d(z, Tz) \quad (\text{by (1)}) \\
 &= (r^2 + \alpha r^2 + \alpha)d(z, Tz) \\
 &= \left\{ r^2 + \left[\frac{r}{1+r} \right] r^2 + \left[\frac{r}{1+r} \right] \right\} d(z, Tz) \quad \left(\text{Q } r = \frac{\alpha}{1-\alpha} \Rightarrow \alpha = \frac{r}{1+r} \right) \\
 &= \left\{ r^2 + \frac{r^3}{1+r} + \frac{r}{1+r} \right\} d(z, Tz) \\
 &= \frac{r^2(1+r) + r^3 + r}{1+r} d(z, Tz) \\
 &= \frac{2r^3 + r^2 + r}{1+r} d(z, Tz) < d(z, Tz) \\
 &\quad \left(\text{Q } \frac{2r^3 + r^2 + r}{1+r} < 1 \text{ if } 0 \leq r \leq \frac{\sqrt{5}-1}{2} \right) \\
 &\quad (\text{See lemma 3.1})
 \end{aligned}$$

Thus $d(z, Tz) < d(z, Tz)$. This is a contradiction.

Case 2. $\frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}}$.

Assume that $d(T^2z, z) < \theta(r)d(T^2z, T^3z)$. Then we have

$$\begin{aligned}
 d(z, Tz) &\leq d(z, T^2z) + d(Tz, T^2z) \quad (\text{by triangle inequality}) \\
 &< \theta(r)d(T^2z, T^3z) + d(Tz, T^2z) \quad (\text{by above assumption}) \\
 &\leq \theta(r)r^2d(z, Tz) + rd(z, Tz) \quad (\text{by (1)}) \\
 &= [\theta(r)r^2 + r]d(z, Tz) \\
 &= \left[\frac{1-r}{r^2}r^2 + r \right] d(z, Tz) \quad \left(\text{Q } \theta(r) = \frac{1-r}{r^2} \text{ in this case} \right) \\
 &= d(z, Tz)
 \end{aligned}$$

Thus $d(z, Tz) < d(z, Tz)$. This is a contradiction. Hence

$$\theta(r)d(T^2z, T^3z) \leq d(T^2z, z).$$

Then by hypothesis we get

$$d(T^3z, Tz) \leq \alpha d(T^2z, T^3z) + \alpha d(z, Tz) \quad (10)$$

Now consider

$$\begin{aligned}
 d(z, Tz) &\leq d(z, T^3z) + d(Tz, T^3z) \quad (\text{by triangle inequality}) \\
 &\leq \alpha r^2d(Tz, z) + \alpha d(T^2z, T^3z) + \alpha d(z, Tz) \quad (\text{by (8) and (10)}) \\
 &\leq \alpha r^2d(Tz, z) + \alpha r^2d(z, Tz) + \alpha d(z, Tz) \quad (\text{by (1)}) \\
 &= [2\alpha r^2 + \alpha]d(z, Tz) \\
 &= \left(2\frac{r}{1+r}r^2 + \frac{r}{1+r} \right) d(z, Tz) \\
 &= \left(\frac{2r^3 + r}{1+r} \right) d(z, Tz) < d(z, Tz) \\
 &\quad \left(\text{Q } \frac{2r^3 + r}{1+r} < 1 \text{ if } \frac{\sqrt{5}-1}{2} \leq r \leq \frac{1}{\sqrt{2}} \right) \\
 &\quad (\text{See lemma 3.1})
 \end{aligned}$$

Thus $d(z, Tz) < d(z, Tz)$. This is a contradiction.

Case 3. $\frac{1}{\sqrt{2}} \leq r < 1$

We note that from theorem 3.1 for any $x, y \in X$ either

$$\theta(r)d(x, Tx) \leq d(x, y) \text{ or}$$

$$\theta(r)d(Tx, T^2x) \leq d(Tx, y).$$

In particular putting $x = u_{2n}$ and $y = z$ we get,

$$\theta(r)d(u_{2n}, u_{2n+1}) \leq d(u_{2n}, z) \text{ or } \theta(r)d(u_{2n+1}, u_{2n+2}) \leq d(u_{2n+1}, z)$$

holds for every $n \in \mathbb{N}$.

By hypothesis we get

$$d(Tu_{2n}, Tz) \leq \alpha d(u_{2n}, Tu_{2n}) + \alpha d(z, Tz) \text{ or}$$

$$d(Tu_{2n+1}, Tz) \leq \alpha d(u_{2n+1}, Tu_{2n+1}) + \alpha d(z, Tz).$$

That is

$$d(u_{2n+1}, Tz) \leq \alpha d(u_{2n}, u_{2n+1}) + \alpha d(z, Tz) \text{ or}$$

$$d(u_{2n+2}, Tz) \leq \alpha d(u_{2n+1}, u_{2n+2}) + \alpha d(z, Tz)$$

holds for every $n \in \mathbb{N}$.

Taking limit as $n \rightarrow \infty$ and using the fact that $u_n \rightarrow z$ as $n \rightarrow \infty$ we get

$$d(z, Tz) \leq \alpha d(z, Tz)$$

$$\therefore d(z, Tz) = 0 \quad (\text{Q } \alpha \in [1/2, 1])$$

$$\therefore Tz = z \quad (\text{Q } d(x, y) = 0 \Rightarrow x = y)$$

This is a contradiction. Therefore in all the cases \exists a $j \in \mathbb{N}$

such that $T^jz = z$. Since $T^n z$ is a Cauchy sequence, we have

$$Tz = z.$$

Thus z is fixed point of T .

Uniqueness: Let z' be another fixed point of T . Thus $Tz' = z'$

From (2) we get

$$d(z', z) = d(Tz', z) \leq rd(z', z)$$

Thus $d(z', z) \leq rd(z', z)$.

Therefore $d(z', z) = 0$ ($\forall r \in [0, 1)$) and so $z' = z$. Hence the fixed point of T is unique.

Example 3.1:

Define a complete metric space X by $X = \{-1, 0, 1\}$ and a mapping T on X by $Tx = 0, \forall x \in X$. Let $\theta(r)$ be the mapping defined in above theorem. Then T satisfy the conditions in above theorem. That is

$$\theta(r)d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq \alpha d(x, Tx) + \alpha d(y, Ty) \forall x, y \in X \text{ and } \alpha \in [0, 1/2)$$

$x = 0$ is the unique fixed point of T .

III. CONCLUSION

In this research paper we have used the non-increasing mapping $\theta(r)$ defined by Suzuki and imposed the Kannan type condition on it to obtain a fixed point theorem.

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